

$$\sum_{k=1}^{\infty} \left(\frac{1}{e}\right)^k k^x \approx \Gamma(x+1)$$

I have this odd fixation on partial sums and most of the identities that litter my notebooks come from them. It is no wonder that I eventually got to this point, so for the first trick, let's find a nice formula for the following:

$$\sum_{k=1}^n k a^k$$

Having worked for some years on functions like this, I now have some rather fruitful methods for locating such nice forms. This method is my favorite and doesn't start off intuitively. (I also try to avoid induction and I like direct proofs.)

$$\begin{aligned} (n+1)a^{n+1} + \sum_{k=0}^n k a^k &= \sum_{k=0}^{n+1} k a^k, && \text{;This is just by definition, so there is nothing magical here} \\ &= 0a^0 + \sum_{k=1}^{n+1} k a^k, && \text{;Just peeling off the first term.} \\ &= \sum_{k=0}^n (k+1)a^{k+1}, && \text{;Reindexing like a pro.} \end{aligned}$$

$$\begin{aligned} (n+1)a^{n+1} &= \sum_{k=0}^n (k+1)a^{k+1} - \sum_{k=0}^n k a^k, && \text{;Shuffling stuff around} \\ &= \sum_{k=0}^n ((k+1)a^{k+1} - k a^k), && \text{;combining the sums (same index)} \\ &= \sum_{k=0}^n (k a^{k+1} + a^{k+1} - k a^k), && \text{;shuffling} \\ &= \sum_{k=0}^n (k a^{k+1} - k a^k) + \sum_{k=0}^n a^{k+1}, && \text{;shuffling} \\ &= \sum_{k=0}^n (k a^{k+1} - k a^k) + \frac{a^{n+1}-a}{a-1}, && \text{;This is something I proved eslewhere, don't worry.} \end{aligned}$$

$$\begin{aligned} (n+1)a^{n+1} - \frac{a^{n+1}-a}{a-1} &= \sum_{k=0}^n (k a^{k+1} - k a^k), && \text{;This is something I proved eslewhere, don't worry.} \\ &= \sum_{k=0}^n k a^k (a-1), && \text{;Factoring out a constant} \\ &= (a-1) \sum_{k=0}^n k a^k, && \text{;Factoring out a constant} \end{aligned}$$

$$\frac{(n+1)a^{n+1}}{a-1} - \frac{a^{n+1}-a}{(a-1)^2} = \sum_{k=0}^n k a^k. \quad \text{;Factoring out a constant}$$

Now you can see that the $k=0$ term is 0, so we can just reindex that to:

$$\frac{(n+1)a^{n+1}}{a-1} - \frac{a^{n+1}-a}{(a-1)^2} = \sum_{k=1}^n k a^k.$$

Tada! Unfortunately, that was the easy one, so I will leave it to the users discretion to derive the

formulas for

$$\sum_{k=1}^n k^b a^k = \frac{(n+1)^b a^{n+1} - \sum_{k=0}^{n-1} (a^{k+1} \sum_{i=0}^{b-1} k^i \binom{b}{i})}{a-1}.$$

(I would have used 'b choose i', but Google Docs doesn't support the nice notation.)

Now let's look at the case where $a \in \mathbb{C}$, $|a| < 1$, because this produces super snazzy results and is a little easier to work with if we take $n \rightarrow \infty$:

$$f_0 = \lim_{n \rightarrow \infty} \sum_{k=1}^n k^0 a^k = \lim_{n \rightarrow \infty} \sum_{k=1}^n a^k = \frac{a}{1-a}$$

$$f_1 = \lim_{n \rightarrow \infty} \sum_{k=1}^n k^1 a^k = \lim_{n \rightarrow \infty} \frac{(n+1)a^{n+1}}{a-1} - \frac{a^{n+1}-a}{(a-1)^2} = \lim_{n \rightarrow \infty} 0 - \frac{0-a}{(a-1)^2} = \frac{a}{(a-1)^2}$$

Again, I will leave it to the reader to derive the following because typing the steps is a pain:

$$f_n = \frac{a}{1-a} \sum_{k=0}^{n-1} f_k \binom{n}{k}$$

Well, that was pleasant, but now we introduce more fun. I like ratios and limits. We will quickly see that $\lim_{n \rightarrow \infty} \frac{f_n}{f_{n-1}} = \infty$, so that's no fun, but the more spectacular thing is that this ratio converges to diverging at a linear rate. Err, what I mean is that if you take the difference of two of these ratios, it converges to a non-zero constant:

$$\lim_{n \rightarrow \infty} \frac{f_{n+1}}{f_n} - \frac{f_n}{f_{n-1}} = c, \quad c \in \mathbb{C}$$

What does this mean? In mathematical term, for $\varepsilon > 0$ there exists a natural number N such that for $n \geq N$, $\left| \frac{f_{n+1}}{f_n} - \frac{f_n}{f_{n-1}} - c \right| < \varepsilon$. But *really* what I am saying is that at some point, the difference of those two ratios is so close to c , we can call it c for all intents and purposes. So if $f_N = x$ and $f_{N+1} = y$, then we know that $f_{N+2} = y(c + \frac{y}{x})$ and then we know that $f_{N+3} = y(2c + \frac{y}{x})(c + \frac{y}{x})$ and in general, $f_{N+k} = y(c + \frac{y}{x})(2c + \frac{y}{x}) \dots ((k-1)c + \frac{y}{x})$

Since f is a function of a , (and continuous) then we should be able to find a value for a such that $c = 1$ which would give us:

$$f_{N+k} = y(c + \frac{y}{x})(2c + \frac{y}{x}) \dots ((k-1)c + \frac{y}{x})$$

$$f_{N+k} = (1 + \frac{y}{x})(2 + \frac{y}{x}) \dots ((k-1) + \frac{y}{x})$$

From this, we see that f is a function that grows at a factorial rate. Without getting too far into the computational mess, $\frac{1}{e}$ is our special value of a that gives us $c = 1$, and as expected, f converges very quickly to the factorial function. It is also pretty clear from our original equation

using the fact that $k > 1$, $k^b < k^{b+\varepsilon}$ for $\varepsilon > 0$ that $\sum_{k=1}^{\infty} k^b a^k$ is a monotone increasing function of b

and is concave up for all b . Therefore, $\sum_{k=1}^{\infty} (\frac{1}{e})^k k^x \approx \Gamma(x+1)$.